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Simple separability for steady heat conduction with spatially-varying thermal conductivity

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INTRODUCTION

IN THE absence of distributed sources, steady-state heat conduction in solids is governed by the partial differential equation

$$\operatorname{div}\cdot(k \operatorname{grad} T) = 0 \tag{1}$$

where k is the thermal conductivity, a physical property of the solid. If the thermal conductivity is approximated as a constant, then equation (1) reduces to the familiar Laplace's equation

$$\nabla^2 T = 0. \tag{2}$$

The problem of solving equation (1) for temperature-dependent thermal conductivity ($k = k(T)$ only) can also effectively be reduced to that of solving Laplace's equation through the application of the Kirchhoff transformation [1]. However, convective boundary conditions may be difficult to account for with classical analytical solution techniques.

One particularly effective analytical technique for solving heat conduction problems governed by Laplace's equation is the method of separation of variables. The formal requirements for simple separability of Laplace's equation in orthogonal, curvilinear coordinate systems have been thoroughly studied and outlined by Stackel [2], Robertson [3], Eisenhart [4] and Moon and Spencer [5].

Unfortunately, relatively little information is available for applying the method of separation of variables to a more general heat conduction problem described by equation (1) with the thermal conductivity varying spatially, or

$$k = k(u_1, u_2, u_3) \tag{3}$$

where (u_1, u_2, u_3) represents an orthogonal, curvilinear coordinate system.

Heat conduction with spatially-varying thermal conductivity can arise in a number of practical applications including thermal contact resistance between dissimilar materials and heat transfer in microelectronic components. The goal of this work is to determine the restrictions on the thermal conductivity such that equation (1) is simply separable whenever Laplace's equation is simply separable and to briefly examine the nature of the solutions which can then be obtained by separation of variables.

THEORETICAL CONSIDERATIONS

For an orthogonal, curvilinear coordinate system, equation (1) can be written as [6]

$$\frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{\partial}{\partial u_i}\left(\frac{\sqrt{g}}{g_i} k \frac{\partial T}{\partial u_i}\right) = 0 \tag{4}$$

where the metric coefficients g_i are defined by the transformation from the Cartesian coordinate system (x, y, z) to the orthogonal, curvilinear coordinate system (u_1, u_2, u_3) such that

$$g_i \equiv \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \tag{5}$$

and

$$g \equiv g_1 g_2 g_3.$$

Definition I. From Moon and Spencer [5, 6], if the assumption

$$T = \prod_{i=1}^{3} U_i(u_i) \tag{6}$$

permits the separation of the partial differential equation into three ordinary differential equations, the equation is said to be simply separable.

Theorem I. The necessary and sufficient condition for simple separability of Laplace's equation in an orthogonal, curvilinear coordinate system with $T = T(u_1, u_2, u_3)$ is that the metric coefficients can be written as

$$\frac{\sqrt{g}}{g_i} = M_{i1} \prod_{j=1}^{3} f_j(u_j) \tag{7}$$

where M_{i1} are cofactors of the first column of the Stackel matrix $[S]$ as thoroughly discussed by Moon and Spencer [6]. The Stackel matrix is of the form

$$[S] = \begin{bmatrix} \Phi_{11}(u_1) & \Phi_{12}(u_1) & \Phi_{13}(u_1) \\ \Phi_{21}(u_2) & \Phi_{22}(u_2) & \Phi_{23}(u_2) \\ \Phi_{31}(u_3) & \Phi_{32}(u_3) & \Phi_{33}(u_3) \end{bmatrix} = [\Phi_{ij}(u_i);\ \ i,j = 1,2,3]. \tag{8}$$

A given coordinate system is then said to permit simple separability of Laplace's equation if a Stackel matrix $[S]$ can be constructed such that equation (7) holds for $i = 1, 2, 3$. Note that the cofactors M_{i1} are not functions of the u_i-coordinate. Both the cofactors M_{i1} and the function $f_i(u_i)$ have been tabulated for most coordinate systems of interest in Moon and Spencer [7]. An excellent proof of Theorem I is contained in Moon and Spencer [6].

CONDITION FOR SIMPLE SEPARABILITY OF EQUATION (4)

Theorem II. Given that Laplace's equation is simply separable in some orthogonal, curvilinear coordinate system, the necessary and sufficient condition for simple separability of the steady heat conduction equation with spatially-varying thermal conductivity in the same coordinate system is that the thermal conductivity can be written as

$$k = k_0 \prod_{i=1}^{3} \omega_i(u_i) \tag{9}$$

where k_0 is an arbitrary constant and the functions $\omega_i(u_i)$ describe the thermal conductivity variation in the u_i-coordinate direction. The functions $\omega_i(u_i)$ and their first derivatives are further assumed to be continuous and bounded.

Proof. By analogy with the work of Robertson [3] and Moon and Spencer [5], a necessary condition for the simple separation of a partial differential equation in the form of

NOMENCLATURE

- a radius of finite cylinder in the example problem
- A_n series coefficients in the example problem
- f_i functions of metric coefficients, $f_i = f_i(u_i)$
- $f(\lambda_n)$ separation constant as a function of the eigenvalues
- g_i metric coefficients given by equation (5), also $g \equiv g_1 g_2 g_3$
- $I_\tau(\cdot)$ modified Bessel function of the first kind of order τ
- $J_0(\cdot)$ Bessel function of the first kind of order zero
- $J_1(\cdot)$ Bessel function of the first kind of order one
- k thermal conductivity, in general $k = k(u_1, u_2, u_3)$
- $K_\tau(\cdot)$ modified Bessel function of the second kind of order τ
- L length of finite cylinder in the example problem
- M_{i1} cofactors of the Stackel matrix $[S]$
- p_i orthogonality weight in Sturm–Liouville problem $p_i = p_i(u_i) = f_i\omega_i\gamma_i$
- q_i arbitrary function in the general Sturm–Liouville problem
- $q(r)$ prescribed heat flux in the example problem
- r polar coordinate
- S Stackel matrix, $[S] = [\Phi_{ij}(u_i);\, i, j = 1, 2, 3]$
- T temperature
- u_i orthogonal, curvilinear coordinates ($i = 1, 2, 3$)
- x, y, z Cartesian coordinates.

Greek symbols

- α_j separation constants ($j = 2, 3$)
- β_{lm} constants for homogeneous boundary conditions ($l, m = 1, 2$)
- γ_i chosen by criterion of equation (21), $\gamma_i \equiv |\Phi_{ij}|$ ($j = 2$ or 3)
- δ_{ij} Kronecker delta $\delta_{ij} = 0$, $i \neq j$, $\delta_{ii} = 1$
- λ_n eigenvalue in the Sturm–Liouville problem; positive roots of $J_0(\lambda_n a) = 0$ for the example problem
- μ, ν parameters describing conductivity variation in the example problem (equation (24))
- τ function of ν, $\tau \equiv (1-\nu)/2$
- ϕ_i^n eigenfunction in the Sturm–Liouville problem
- Φ_{ij} element of the Stackel matrix $[S]$, $\Phi_{ij} = \Phi_{ij}(u_i)$ ($i, j = 1, 2, 3$)
- ω_i function which describes the thermal conductivity variation in the u_i-coordinate direction, $\omega_i = \omega_i(u_i)$
- Ω_n function in the example problem given by equation (32).

Subscripts and superscripts

- i usually refers to the u_i-coordinate direction ($i = 1, 2, 3$)
- n usually refers to an eigenvalue λ_n ($n = 1, 2, 3, \ldots$).

equation (4) is evidently that

$$\frac{\sqrt{g}}{g_i} k = F_i(u_i) G_i \tag{10}$$

where $F_i(u_i)$ is a function of the u_i-coordinate only and G_i is a function only of coordinates other than u_i.

Clearly equation (10) is satisfied only when the thermal conductivity is given by equation (9) and thus this restriction represents a necessary condition for simple separability of equation (4).

To show that the form of the thermal conductivity variation given by equation (9) is a sufficient condition for simple separability, the substitution of equation (6), (7) and (9) into equation (4) must produce three ordinary differential equations as required by Definition 1. Direct substitution gives

$$\sum_{i=1}^{3} M_{i1}\left[\prod_{l=1}^{3}(1-\delta_{li})f_l\right]\left[\prod_{m=1}^{3}(1-\delta_{mi})\omega_m\right] \times\left[\prod_{n=1}^{3}(1-\delta_{ni})U_n\right]\left\{\frac{\mathrm{d}}{\mathrm{d}u_i}\left[f_i\omega_i\frac{\mathrm{d}U_i}{\mathrm{d}u_i}\right]\right\} = 0 \tag{11}$$

where δ_{ij} is the familiar Kronecker delta.

Division of all three terms in equation (11) by $T k f_1 f_2 f_3$ gives

$$\sum_{i=1}^{3} M_{i1}\left\{\frac{1}{f_i\omega_i U_i}\frac{\mathrm{d}}{\mathrm{d}u_i}\left[f_i\omega_i\frac{\mathrm{d}U_i}{\mathrm{d}u_i}\right]\right\} = 0. \tag{12}$$

It appears at this point that simple separation has been effected because for M_{i1} neither a function of u_i nor zero, then

$$\frac{\mathrm{d}}{\mathrm{d}u_i}\left[f_i\omega_i\frac{\mathrm{d}U_i}{\mathrm{d}u_i}\right] = 0, \quad i = 1, 2, 3. \tag{13}$$

However, no separation constants have yet been introduced and because U_i can be found by direct integration, equation (13) does not really represent three ordinary differential equations as required by Definition I.

Consider now two arbitrary separation constants α_2 and α_3. From the theory of determinants

$$\sum_{i=1}^{3} \Phi_{i2} M_{i1} = 0 \tag{14}$$

$$\sum_{i=1}^{3} \Phi_{i3} M_{i1} = 0 \tag{15}$$

or thus

$$\sum_{j=2}^{3} \alpha_j \sum_{i=1}^{3} \Phi_{ij} M_{i1} = 0$$

and

$$\sum_{j=2}^{3} M_{i1} \sum_{j=2}^{3} \alpha_j \Phi_{ij} = 0 \tag{16}$$

where again Φ_{ij} are elements of the Stackel matrix $[S]$ and M_{i1} are cofactors of $[S]$.

The addition of equations (12) and (16) gives

$$\sum_{i=1}^{3} M_{i1}\left\{\frac{1}{f_i\omega_i U_i}\frac{\mathrm{d}}{\mathrm{d}u_i}\left[f_i\omega_i\frac{\mathrm{d}U_i}{\mathrm{d}u_i}\right] + \sum_{j=2}^{3}\alpha_j\Phi_{ij}\right\} = 0. \tag{17}$$

Since again the cofactors M_{i1} are neither a function of u_i nor zero, equation (17) reduces to

$$\frac{1}{f_i}\frac{\mathrm{d}}{\mathrm{d}u_i}\left(f_i\frac{\mathrm{d}U_i}{\mathrm{d}u_i}\right) + \frac{1}{\omega_i}\frac{\mathrm{d}\omega_i}{\mathrm{d}u_i}\frac{\mathrm{d}U_i}{\mathrm{d}u_i} + U_i\sum_{j=2}^{3}\alpha_j\Phi_{ij} = 0, \quad i = 1, 2, 3 \tag{18}$$

which represents three ordinary differential equations for U_i as required by Definition I. Thus a thermal conductivity variation of the form of equation (9) represents a sufficient condition for simple separability of equation (4) and hence Theorem II is proved.

PRACTICAL CONSIDERATIONS

Since the functions $f_i(u_i)$ and $\Phi_{ij}(u_i)$ are tabulated in Moon and Spencer [7] for most coordinate systems of interest, the separation of the heat conduction equation with spatially-varying thermal conductivity requires only direct substitution into equation (18). The choice of appropriate separation constants is made by examining the homogeneous boundary conditions.

The single non-homogeneous boundary condition in a well-posed problem is accounted for with the method of separation of variables by applying the orthogonality of eigenfunction solutions to the separated ordinary differential equations. If for some coordinate direction $u_i^l \leq u_i \leq u_i^r$ the homogeneous boundary conditions are of the form

$$\beta_{11} U_i(u_i^l) - \beta_{12} \frac{dU_i}{du_i}(u_i^l) = 0$$
$$\beta_{21} U_i(u_i^r) + \beta_{22} \frac{dU_i}{du_i}(u_i^r) = 0 \tag{19}$$

then comparison with the general Sturm–Liouville problem [6] gives an orthogonality weight

$$p_i(u_i) = f_i \omega_i \gamma_i \tag{20}$$

where $\gamma_i \equiv |\Phi_{ij}| > 0$ is chosen such that for either $j = 2$ or 3

$$\alpha_j \Phi_{ij} = f(\lambda_n)\gamma_i > 0. \tag{21}$$

The λ_n are the eigenvalues of the Sturm–Liouville problem

$$\frac{1}{f_i}\frac{d}{du_i}\left(f_i \frac{d\phi_i^n}{du_i}\right) + \frac{1}{\omega_i}\frac{d\omega_i}{du_i}\frac{d\phi_i^n}{du_i} + (q_i(u_i) + f(\lambda_n)\gamma_i)\phi_i^n = 0 \tag{22}$$

where $\phi_i^n(u_i)$ is subject to the homogeneous boundary conditions of equation (19).

The orthogonality relationship for the eigenfunctions $\phi_i^n(u_i)$ is then

$$\int_{u_i^l}^{u_i^r} \phi_i^n \phi_i^m f_i \omega_i \gamma_i \, du_i = 0 \quad \text{for} \quad m \neq n. \tag{23}$$

With the orthogonality relationship of equation (23) complete solutions to steady heat conduction problems can be constructed without actually knowing beforehand how to compute the eigenvalues λ_n or the eigenfunctions $\phi_i^n(u_i)$. A combination of Chebyshev methods [8] and asymptotic analysis [9] may be required to compute the λ_n and $\phi_i^n(u_i)$ for the potentially complex but linear separated ordinary differential equations encountered with spatially-varying thermal conductivity.

As with the Kirchhoff transformation for temperature-dependent thermal conductivity, only Dirichlet and Neumann boundary conditions can be accounted for with the classical method of separation of variables applied to problems of spatially-varying thermal conductivity. For many coordinate systems this is not an extra restriction since the boundary condition of the third kind or the Robin condition cannot usually be applied where the metric coefficients g_i are functions of coordinates other than u_i. Furthermore, if a problem with spatially-varying thermal conductivity does have one or more convective boundaries, the simple separability of the temperature distribution can still exist and the unknown series coefficients in the solution can be determined by the methods outlined by Negus and Yovanovich [10].

EXAMPLE PROBLEM

Consider the problem shown in Fig. 1 where a prescribed heat flux $q(r)$ enters the top surface of a finite cylinder and exits at the side surface which is maintained at $T = 0$. The thermal conductivity in the cylinder is assumed to obey the power-law relationship

$$k = k_0(z+\mu)^\nu \tag{24}$$

where $z+\mu > 0$ for $0 < z < L$.

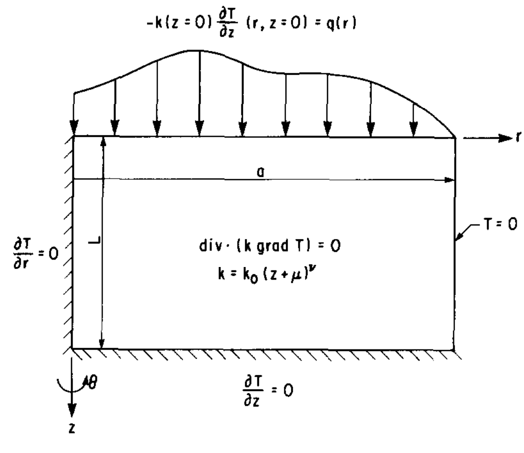

FIG. 1. Example problem with spatially-varying thermal conductivity in a finite cylinder.

If this problem geometry is compared with the circular cylindrical coordinate system of Moon and Spencer [7] then

$$u_1 \equiv r; \quad u_2 \equiv \theta; \quad u_3 \equiv z.$$

For a temperature distribution independent of θ, the separation constants are $\alpha_2 = 0$ and $\alpha_3 = \pm\lambda^2, 0$. Consideration of the homogeneous boundary conditions at $r = 0$ and a shows that the separation cases $\alpha_3 = \lambda^2$ and 0 produce trivial solutions. For $\alpha_3 = -\lambda^2$ the separated equations are

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0 \tag{25}$$

$$\frac{d^2Z}{dz^2} + \frac{\nu}{z+\mu}\frac{dZ}{dz} - \lambda^2 Z = 0 \tag{26}$$

where $T(r,z) = R(r)Z(z)$. These separated equations are obtained by substituting the following information from Moon and Spencer [7] and equation (24) into equation (18):

$$f_1 = r; \qquad f_3 = 1$$
$$\Phi_{13} = -1; \quad \Phi_{33} = 1$$
$$\omega_1 = 1; \qquad \omega_3 = (z+\mu)^\nu.$$

The general solution to equation (25) is given in terms of Bessel functions of order zero. After applying the homogeneous boundary conditions

$$\frac{dR}{dr}(r=0) = 0; \quad R(r=a) = 0$$

the function $R(r)$ is given by

$$R(r) = \sum_{n=1}^{\infty} B_n J_0(\lambda_n r) \tag{27}$$

where $J_0(\cdot)$ is the Bessel function of the first kind of order zero and λ_n are the positive roots of

$$J_0(\lambda_n a) = 0, \quad n = 1, 2, 3, \ldots \tag{28}$$

The solution of equation (26) can be found by comparing with a more general form of Bessel's equation [11] to give

$$Z(z) = (z+\mu)^\tau[CI_\tau(\lambda_n(z+\mu)) + DK_\tau(\lambda_n(z+\mu))] \tag{29}$$

where

$$\tau \equiv (1-\nu)/2 \tag{30}$$

and $I_\tau(\cdot)$ and $K_\tau(\cdot)$ are modified Bessel functions of the first and second kinds, respectively, of order τ.

After applying the homogeneous boundary condition $dZ/dz = 0$ at $z = L$, the temperature distribution in the finite cylinder is found to be

$$T(r,z) = (z+\mu)^{\tau} \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)[I_\tau(\lambda_n(z+\mu)) + \Omega_n K_\tau(\lambda_n(z+\mu))] \quad (31)$$

where

$$\Omega_n \equiv \frac{I_{\tau-1}(\lambda_n(L+\mu))}{K_{\tau-1}(\lambda_n(L+\mu))}. \quad (32)$$

The unknown coefficients A_n are determined by considering the non-homogeneous boundary condition on $z = 0$ where

$$-k(z=0)\frac{\partial T}{\partial z}(r, z=0) = q(r), \quad 0 \leqslant r \leqslant a. \quad (33)$$

Differentiation of equation (31) and substitution into equation (33) then gives

$$q(r) = k_0 \mu^{\tau+\nu} \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)\lambda_n[\Omega_n K_{\tau-1}(\lambda_n \mu) - I_{\tau-1}(\lambda_n \mu)]. \quad (34)$$

If both sides of equation (34) are multiplied by $J_0(\lambda_m r)\,\mathrm{d}r$ and integrated from $r = 0$ to a, then by the orthogonality of the eigenfunctions $J_0(\lambda_n r)$ on the interval $0 \leqslant r \leqslant a$ the unknown coefficients are given by

$$A_n = \frac{2\int_0^a q(r) J_0(\lambda_n r) r\,\mathrm{d}r}{k_0 \mu^{\tau+\nu} \lambda_n a^2 J_1^2(\lambda_n a)[\Omega_n K_{\tau-1}(\lambda_n \mu) - I_{\tau-1}(\lambda_n \mu)]} \quad (35)$$

where $J_1(\cdot)$ is the Bessel function of the first kind of order one.

In summary the temperature distribution in the finite cylinder of Fig. 1 with spatially-varying thermal conductivity described by equation (24) is given exactly by equation (31) combined with equations (28), (30), (32) and (35). This example problem illustrates that constructing the solution to a steady heat conduction problem with spatially-varying thermal conductivity by the method of separation of variables is conceptually similar to solving Laplace's equation by separation of variables. The major difference is that both the mathematical details of the analysis and the subsequent computation of the temperature field can be potentially much more complex with spatially-varying thermal conductivity.

CONCLUSIONS

A necessary and sufficient criterion has been developed to determine the conditions for which the steady heat conduction equation with spatially-varying thermal conductivity is simply separable given that Laplace's equation is known to be simply separable in some chosen orthogonal, curvilinear coordinate system. The orthogonality relationship for the separated ordinary differential equations associated with spatially-varying thermal conductivity has also been examined.

An important corollary to the restriction on the thermal conductivity imposed by Theorem II is that a one-dimensional variation, regardless of its complexity, always permits solution by simple separation of variables assuming Laplace's equation is also simply separable in the chosen coordinate system.

In practice the restriction on the thermal conductivity imposed by Theorem II combined with the tedium of the method of separation of variables, especially for three-dimensional problems, may reduce the number of applications in which the solution methods of this work will be favoured. However, exact solutions for spatially-varying thermal conductivity can still provide a method for validating the operation of other approximate or numerical solution techniques.

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